

The numerical range and the spectrum of a product of two orthogonal projections

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Abstract

The aim of this paper is to describe the closure of the numerical range of the product of two orthogonal projections in Hilbert space as a closed convex hull of some explicit ellipses parametrized by points in the spectrum. Several improvements (removing the closure of the numerical range of the operator, using a parametrization after its eigenvalues) are possible under additional assumptions. An estimate of the least angular opening of a sector with vertex 1 containing the numerical range of a product of two orthogonal projections onto two subspaces is given in terms of the cosine of the Friedrichs angle. Applications to the rate of convergence in the method of alternating projections and to the uncertainty principle in harmonic analysis are also discussed.

Keywords: Numerical range; orthogonal projections; Friedrich angle; method of alternating projections; uncertainty principle; annihilating pair.

1 Introduction

Background. The numerical range of a Hilbert space operator $T \in \mathcal{B}(H)$ is defined as $W(T) = \{\langle Tx, x \rangle, x \in H, \|x\| = 1\}$. It is always a convex set in the complex plane (the Toeplitz-Hausdorff theorem) containing in its closure the spectrum of the operator. Also, the intersection of the closure of the numerical ranges of all the operators similar to T is precisely the convex hull of the spectrum of T (Hildebrandt's theorem). We refer to the book [GR97] for these and other facts about numerical ranges. Another useful property the numerical ranges have is the following recent result of Crouzeix [Cro07]: for every $T \in \mathcal{B}(H)$ and every polynomial p , we have $\|p(T)\| \leq 12 \sup_{z \in W(T)} |p(z)|$.

The problem. The main aim of this paper is to study the numerical range $W(T)$ and the numerical radius, defined by $\omega(T) = \sup\{|z|, z \in W(T)\}$, of a product of two orthogonal projections $T = P_{M_2}P_{M_1}$. In what follows we denote by P_M the orthogonal projection onto the closed subspace M of a given Hilbert space H . We prove a representation of the closure of $W(T)$ as a closed convex hull of some explicit ellipses parametrized by points in the spectrum $\sigma(T)$ of T and we discuss several applications. We also study the relationship between the numerical range (numerical radius) of a product of two orthogonal projections and its spectrum (resp. spectral radius). Recall that the spectral radius $r(T)$ of $T \in \mathcal{B}(H)$ is defined as $r(T) = \sup\{|z|, z \in \sigma(T)\}$.

Previous results. Orthogonal projections in Hilbert space are basic objects of study in Operator theory. Products or sums of orthogonal projections, in finite or

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infinite dimensional Hilbert spaces, appear in various problems and in many different areas, pure or applied. We refer the reader to a book [Gal04] and two recent surveys [Gal08, BS10] for more information. The fact that the numerical range of a finite product of orthogonal projections is included in some sector of the complex plane with vertex at 1 was an essential ingredient in the proof by Delyon and Delyon [DD99] of a conjecture of Burkholder, saying that the iterates of a product of conditional expectations are almost surely convergent to some conditional expectation in an L^2 space (see also [Cro08, Coh07]). For a product of two orthogonal projections we know that the numerical range is included in a sector with vertex one and angle $\pi/6$ ([Cro08]).

The spectrum of a product of two orthogonal projections appears naturally in the study of the rate of convergence in the strong operator topology of $(P_{M_2}P_{M_1})^n$ to $P_{M_1 \cap M_2}$ (cf. [Deu01, BDH09, DH10a, DH10b, BGM, BGM10, BL10]). This is a particular instance of von Neumann-Halperin type theorems, sometimes called in the literature the method of alternating projections. The following dichotomy holds (see [BDH09]): either the sequence $(P_{M_2}P_{M_1})^n$ converge uniformly with an exponential speed to $P_{M_1 \cap M_2}$ (if $1 \notin \sigma(P_{M_2}P_{M_1})$), or the sequence of alternating projections $(P_{M_2}P_{M_1})^n$ converges arbitrarily slowly in the strong operator topology (if $1 \in \sigma(P_{M_2}P_{M_1})$). We refer to [BGM, BGM10] for several possible meanings of “slow convergence”.

An occurrence of the numerical range of operators related to sums of orthogonal projections appears also in some Harmonic analysis problems. The uncertainty principle in Fourier analysis is the informal assertion that a function $f \in L_2(\mathbb{R})$ and its Fourier transform $\mathcal{F}(f)$ cannot be too small simultaneously. Annihilating pairs and strong annihilating pairs are a way to formulate this idea (precise definitions will be given in Section 5). Characterizations of annihilating pairs and strong annihilating pairs (S, Σ) in terms of the numerical range of the operator $P_S + iP_\Sigma$, constructed using some associated orthogonal projections P_S and P_Σ , can be found in [HJ94, Len72].

Main Results. Our first contribution is an exact formula for the closure of the numerical range $\overline{W(P_{M_2}P_{M_1})}$, expressed as a convex hull of some ellipses $\mathcal{E}(\lambda)$, parametrized by points in the spectrum ($\lambda \in \sigma(P_{M_2}P_{M_1})$).

Definition 1.1. Let $\lambda \in [0, 1]$. We denote $\mathcal{E}(\lambda)$ the domain delimited by the ellipse with foci 0 and λ , and minor axis length $\sqrt{\lambda(1-\lambda)}$.

We refer to Remark 3.3 and to Figure 1 for more information about these ellipses.

Theorem 1.2. Let M_1 and M_2 be two closed subspaces of H such that $M_1 \neq H$ or $M_2 \neq H$. Then the closure of the numerical range of $P_{M_2}P_{M_1}$ is the closure of the convex hull of the ellipses $\mathcal{E}(\lambda)$ for $\lambda \in \sigma(P_{M_2}P_{M_1})$, i.e.:

$$\overline{W(P_{M_2}P_{M_1})} = \overline{\text{conv}_{\lambda \in \sigma(P_{M_2}P_{M_1})} \{\mathcal{E}(\lambda)\}}.$$

The proof uses in an essential way Halmos’ two subspaces theorem recalled in the next section. We will use a completely different approach to describe the numerical range (without the closure) of $T = P_{M_2}P_{M_1}$ under the additional assumption that the self-adjoint operator $T^*T = P_{M_1}P_{M_2}P_{M_1}$ is diagonalisable (see Definition 3.7). In this case the numerical range $W(T)$ is the convex hull of the same ellipses as before but this time parametrized by the point spectrum $\sigma_p(T)$ (=eigenvalues) of $T = P_{M_2}P_{M_1}$.

Theorem 1.3. Let H be a separable Hilbert space. Let M_1 and M_2 be two closed subspaces of a Hilbert space H such that $M_1 \neq H$ or $M_2 \neq H$. If $P_{M_1}P_{M_2}P_{M_1}$ is diagonalizable, then the numerical range $W(P_{M_2}P_{M_1})$ is the convex hull of the ellipses $\mathcal{E}(\lambda)$, with the λ ’s being the eigenvalues of $P_{M_2}P_{M_1}$, i.e.:

$$W(P_{M_2}P_{M_1}) = \text{conv}_{\lambda \in \sigma_p(P_{M_2}P_{M_1})} \{\mathcal{E}(\lambda)\}.$$

Concerning the relationship between the numerical radius and the spectral radius of a product of two orthogonal projections we prove the following result.

Proposition 1.4. *Let M_1, M_2 be two closed subspaces of H . The numerical radius and the spectral radius of $P_{M_2}P_{M_1}$ are linked by the following formula:*

$$\omega(P_{M_2}P_{M_1}) = \frac{1}{2} \left(\sqrt{r(P_{M_2}P_{M_1})} + r(P_{M_2}P_{M_1}) \right).$$

The proof is an application of Theorem 1.2 and the obtained formula is better than Kittaneh's inequality [Kit03] whenever the Friedrichs angle (Definition 2.6) between M_1 and M_2 is positive.

Theorems 1.2 and 1.3 can be used to localize $W(P_{M_2}P_{M_1})$ even if the spectrum of $P_{M_2}P_{M_1}$ is unknown. We mention here the following important consequence about the inclusion of $W(P_{M_2}P_{M_1})$ in a sector of vertex 1 whose angular opening is expressed in terms of the cosine of the Friedrichs angle $\cos(M_1, M_2)$ between the subspaces M_1 and M_2 . This is a refinement of the Crouzeix's result [Cro08] for products of two orthogonal projections.

Proposition 1.5. *Let M_1 and M_2 be two closed subspaces of a Hilbert space H . We have the following inclusion:*

$$W(P_{M_2}P_{M_1}) \subset \left\{ z \in \mathbb{C}, |\arg(1 - z)| \leq \arctan\left(\sqrt{\frac{\cos^2(M_1, M_2)}{4 - \cos^2(M_1, M_2)}}\right) \right\}.$$

We next consider some inverse spectral problems and construct examples of projections such that the spectrum of their product is a prescribed compact set included in $[0, 1]$. These examples will generalise to the infinite dimensional setting a result due to Nelson and Neumann [NN87]. We will also give examples that answers two open questions stated in a article of Nees [Nee99].

The following result allows to find $\sigma(P_{M_2}P_{M_1}) \cap [\frac{1}{4}, 1]$, the points of the spectrum which are larger than $1/4$, whenever the closure $\overline{W(P_{M_2}P_{M_1})}$ of the numerical range is known.

Theorem 1.6. *Let $\alpha \in [\frac{\pi}{3}, \pi]$. The following assertions are equivalent:*

1. $\frac{1}{2(1-\cos(\alpha))} \in \sigma(P_{M_2}P_{M_1})$;
2. $\sup\{\operatorname{Re}(z \exp(-i\alpha)), z \in W(P_{M_2}P_{M_1})\} = \frac{1}{4(1-\cos(\alpha))}$.

Actually it is possible to obtain a description of the entire spectrum $\sigma(P_{M_2}P_{M_1})$ starting from $\overline{W(P_{M_2}P_{M_1})}$ and $\overline{W(P_{M_2}(I - P_{M_1}))}$.

Finally, we will explain how the relation $1 \in W(P_{M_2}P_{M_1})$ is related to arbitrarily slow convergence in the von Neumann-Halperin theorem and we will give new characterizations of annihilating pairs and strong annihilating pairs in terms of $W(P_S P_\Sigma)$.

Organization of the paper. The rest of the paper is organized as follows. We recall in Section 2 several preliminary notions and known facts that will be useful in the sequel. In Section 3 we discuss the results concerning the exact computation of the numerical range of a product T of two orthogonal projections assuming that the spectrum, or the point spectrum, of T is known. Then we will give some “localization” results about the numerical range of T that require less informations about the spectrum of T . Several examples are also given, some of them leading to an answer of two open questions from [Nee99]. In Section 4 we discuss the inverse problem of describing the spectrum of T knowing its numerical range, and the relationship between the numerical and spectral radii of T . The paper ends with two applications of these results, one concerning the rate of convergence in the method of alternating projections and the second one concerning the uncertainty principle.

2 Preliminaries

In this section we introduce some notation and recall several useful facts and results.

2.1 Halmos' two subspaces theorem

For a fixed Hilbert space H and a closed subspace M of H we denote by M^\perp the orthogonal complement of M in H and by P_M the orthogonal projection onto M . Let now M_1 and M_2 be two closed subspaces of a Hilbert space H . Consider the following orthogonal decomposition:

$$H = (M_1 \cap M_2) \oplus (M_1 \cap M_2^\perp) \oplus (M_1^\perp \cap M_2) \oplus (M_1^\perp \cap M_2^\perp) \oplus \tilde{H}, \quad (1)$$

where \tilde{H} is the orthogonal complement of the first 4 subspaces. With respect to this orthogonal decomposition we can write:

$$\begin{aligned} P_{M_1} &= I \oplus 0 \oplus 0 \oplus 0 \oplus \tilde{P}_1 \\ P_{M_2} &= I \oplus 0 \oplus I \oplus 0 \oplus \tilde{P}_2 \\ P_{M_2} P_{M_1} &= I \oplus 0 \oplus 0 \oplus 0 \oplus \tilde{P}_2 \tilde{P}_1. \end{aligned}$$

Using the formula $W(T \oplus S) = \text{conv}\{W(T), W(S)\}$ (see for instance [GR97]) we have $W(P_{M_2} P_{M_1}) = \text{conv}\{\{1\}, \{0\}, W(\tilde{P}_2 \tilde{P}_1)\}$ whenever the corresponding subspaces $M_1^{(\perp)} \cap M_2^{(\perp)}$ are not equal to $\{0\}$.

Definition 2.1. Let N_1, N_2 be two closed subspaces of an Hilbert space H . We say that (N_1, N_2) are in *generic position* if:

$$N_1 \cap N_2 = N_1^\perp \cap N_2 = N_1 \cap N_2^\perp = N_1^\perp \cap N_2^\perp = \{0\}.$$

In Sections 2 and 3 we will denote pairs of subspaces in generic position by (N_1, N_2) , in order to distinguish them from pairs of general closed subspaces (M_1, M_2) .

We say that A is unitary equivalent to B (and write $A \sim B$) if there exists a unitary operator U such that $A = UBU^*$. The following result, Halmos' two subspace theorem [Hal69], is a useful description of orthogonal projections of two subspaces in generic position.

Theorem 2.2. *If (N_1, N_2) are in generic position, then there exists a subspace K of H such that H is unitary equivalent to $K \oplus K$. Also, there exist two operators $C, S \in \mathcal{B}(K)$ such that $0 \leq C \leq I$, $0 \leq S \leq I$ and $C^2 + S^2 = I$, and such that P_1 and P_2 are simultaneously unitary equivalent to the following operators:*

$$P_1 \sim \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, P_2 \sim \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix}.$$

Moreover, there exists an operator T verifying $0 \leq T \leq \frac{\pi}{2}I$ such that $\cos(T) = C$ and $\sin(T) = S$.

For a historical discussion and several applications of Halmos' two subspace theorem we refer the reader to [BS10].

2.2 Support functions

The notion of support functions is classical in convex analysis.

Definition 2.3. Let \mathcal{S} be a bounded convex set in \mathbb{C} . Let $\alpha \in \mathbb{R}$. The support function of \mathcal{S} , of angle α , is defined by the following formula:

$$\rho_{\mathcal{S}}(\alpha) = \sup\{\operatorname{Re}(z \exp(-i\alpha)), z \in \mathcal{S}\}.$$

The following proposition shows that the support function characterizes the closure of convex sets.

Proposition 2.4. We denote by $\overline{\mathcal{S}}$ the closure of \mathcal{S} . We have:

$$\overline{\mathcal{S}} = \{z \in \mathbb{C}, \forall \alpha, \operatorname{Re}(z \exp(-i\alpha)) \leq \rho_{\mathcal{S}}(\alpha)\}.$$

We will need in this paper the following result about support functions.

Lemma 2.5. Let $\mathcal{S}_1, \mathcal{S}_2$ be two bounded convex sets of the plane with support functions $\rho_{\mathcal{S}_1}(\alpha)$ and, respectively, $\rho_{\mathcal{S}_2}(\alpha)$. Let \mathcal{S} be such that $\rho_{\mathcal{S}}(\alpha) = \max_{i=1,2} \rho_{\mathcal{S}_i}(\alpha)$. Then we have $\overline{\mathcal{S}} = \overline{\operatorname{conv}\{\mathcal{S}_1, \mathcal{S}_2\}}$.

A proof of the above propositions and more information about support functions are available in [Roc70].

2.3 Cosine of Friedrichs angle of two subspaces

We now introduce the cosine of the Friedrichs angle between two subspaces. We refer to [Deu01] as a source for more information.

Definition 2.6. Let M_1, M_2 be two closed subsaces of H , with intersection $M = M_1 \cap M_2$. We define the cosine of the Friederichs angle between M_1 and M_2 by the following formula:

$$\cos(M_1, M_2) = \sup\{|\langle x, y \rangle|, x \in M_1 \cap M^\perp, y \in M_2 \cap M^\perp, \|x\| = \|y\| = 1\}.$$

An equivalent way ([KW88, Deu01]) to express the above cosine is given by the formula $\cos^2(M_1, M_2) = \|P_{M_1} P_{M_2} P_{M_1} - P_{M_1 \cap M_2}\|$. The following result, which will be helpful later on, offers a spectral interpretation of $\cos(M_1, M_2)$.

Lemma 2.7. Let M_1 and M_2 be two closed subspaces of H . Then

$$\cos(M_1, M_2) = \sup\{\sqrt{\lambda} : \lambda \in \sigma(P_{M_2} P_{M_1}) \setminus \{1\}\}.$$

This result can be seen as a consequence of Halmos' two subspace theorem (see [BS10]). We present here a different proof.

Proof. We start by remarking that $\sigma(P_{M_2} P_{M_1})$ is a compact subset of $[0, 1]$. Indeed, we have $\sigma(P_{M_1} P_{M_2} P_{M_1}) \setminus \{0\} = \sigma((P_{M_2} P_{M_1}) P_{M_1}) \setminus \{0\} = \sigma(P_{M_2} P_{M_1}) \setminus \{0\}$ and $P_{M_1} P_{M_2} P_{M_1}$ is a self-adjoint operator which is positive and of norm one. Using the decomposition $H = (M_1 \cap M_2) \oplus (M_1 \cap M_2)^\perp$ we can write $P_{M_1} P_{M_2} P_{M_1} = P_{M_1 \cap M_2} \oplus (P_{M_1} P_{M_2} P_{M_1} - P_{M_1 \cap M_2})$, so we get $\sigma(P_{M_1} P_{M_2} P_{M_1}) = \sigma(P_{M_1 \cap M_2}) \cup \sigma(P_{M_1} P_{M_2} P_{M_1} - P_{M_1 \cap M_2})$. Since

$$\cos^2(M_1, M_2) = \|P_{M_1} P_{M_2} P_{M_1} - P_{M_1 \cap M_2}\| = \sup \sigma(P_{M_1} P_{M_2} P_{M_1} - P_{M_1 \cap M_2})$$

we obtain

$$\cos^2(M_1, M_2) = \sup \sigma(P_{M_1} P_{M_2} P_{M_1}) \setminus \{1\} = \sup \sigma(P_{M_2} P_{M_1}) \setminus \{1\}.$$

□

3 Description of the numerical range knowing the spectrum

3.1 The closure of the numerical range as a convex hull of ellipses

The goal of this section is to prove Theorem 1.2 using a description of the support function of $W(P_2P_1)$, which is a closed convex set of \mathbb{C} . This idea appeared for instance in [Len72] in a different context. We will first assume that we are in generic position; the general case will be easily deduced from this particular one.

Lemma 3.1. *Suppose that (N_1, N_2) is in generic position. Then the support function of the numerical range of P_2P_1 is:*

$$\rho_{W(P_2P_1)}(\alpha) = \sup_{\lambda \in \sigma(P_2P_1)} \frac{1}{2}(\cos(\alpha)\lambda + \sqrt{\lambda(1 - \sin(\alpha)^2\lambda)}).$$

Proof. We have that

$$\begin{aligned} \rho_{W(P_2P_1)}(\alpha) &= \sup\{\operatorname{Re}(\langle P_2P_1h, h \rangle \exp(-i\alpha)), h \in H, \|h\| = 1\} \\ &= \sup\{\operatorname{Re}(\langle \exp(-i\alpha)P_2P_1h, h \rangle), h \in H, \|h\| = 1\} \\ &= \sup\{\langle \operatorname{Re}(\exp(-i\alpha)P_2P_1)h, h \rangle, h \in H, \|h\| = 1\}. \end{aligned}$$

Applying Halmos' two subspace theorem, we get

$$P_2P_1 \sim \begin{pmatrix} C^2 & 0 \\ CS & 0 \end{pmatrix}, P_1P_2 \sim \begin{pmatrix} C^2 & CS \\ 0 & 0 \end{pmatrix}.$$

So we have that

$$\operatorname{Re}(\exp(-i\alpha)P_2P_1) \sim \begin{pmatrix} \cos(\alpha)C^2 & \frac{\exp(i\alpha)}{2}CS \\ \frac{\exp(-i\alpha)}{2}CS & 0 \end{pmatrix}.$$

We set

$$M(t, \alpha) = \begin{pmatrix} \cos(\alpha)\cos(t)^2 & \frac{\exp(i\alpha)}{2}\cos(t)\sin(t) \\ \frac{\exp(-i\alpha)}{2}\cos(t)\sin(t) & 0 \end{pmatrix}.$$

Then we have that $\operatorname{Re}(\exp(-i\alpha)P_2P_1) \sim M(T, \alpha)$. As $M(t, \alpha)$ is hermitian, if we denote $v_1(t, \alpha)$ and $v_2(t, \alpha)$ the eigenvalues of $M(t, \alpha)$, we get that

$$M(t, \alpha) \sim \begin{pmatrix} v_1(t, \alpha) & 0 \\ 0 & v_2(t, \alpha) \end{pmatrix}.$$

It can be computed that $v_1(t, \alpha) = \frac{1}{2}(\cos(\alpha)\cos(t)^2 + \cos(t)\sqrt{1 - \sin(\alpha)^2\cos(t)^2})$ and $v_2(t, \alpha) = \frac{1}{2}(\cos(\alpha)\cos(t)^2 - \cos(t)\sqrt{1 - \sin(\alpha)^2\cos(t)^2})$. We have $v_1(t, \alpha) \geq 0$ and $v_2(t, \alpha) \leq 0$ for every $t \in [0, \frac{\pi}{2}]$. As $\sigma(T) \subset [0, \frac{\pi}{2}]$, we obtain the following order relations $v_2(T, \alpha) \leq 0 \leq v_1(T, \alpha)$. This implies $M(T, \alpha) = v_1(T, \alpha) \oplus v_2(T, \alpha)$. Moreover, the operators $v_i(T, \alpha)$ are self-adjoint, $i = 1, 2$. Therefore

$$\begin{aligned} \rho_{W(P_2P_1)}(\alpha) &= \sup_{\|h\|=1} \langle \operatorname{Re}(\exp(-i\alpha)P_2P_1)h, h \rangle \\ &= \sup_{\|x\|=1} \langle v_1(T, \alpha)x, x \rangle \\ &= \|v_1(T, \alpha)\| \\ &= \sup_{t_0 \in \sigma(T)} v_1(t_0, \alpha). \end{aligned}$$

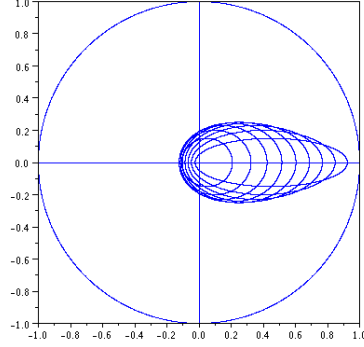


Figure 1: Ellipse $\mathcal{E}(\lambda)$ for $\lambda = 0.1, 0.2, \dots, 0.9$

Halmos' theorem implies that

$$P_1 P_2 P_1 \sim \begin{pmatrix} \cos(T)^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

We have $\sigma(P_2 P_1) \setminus \{0\} = \sigma((P_2 P_1) P_1) \setminus \{0\} = \sigma(P_1 P_2 P_1) \setminus \{0\}$, and $\cos^2(\sigma(T)) \cup \{0\} = \sigma(P_1 P_2 P_1)$. Denoting $\lambda = \cos(t)^2$ and $\tilde{v}_i(\lambda, \alpha) = \frac{1}{2}(\cos(\alpha)\lambda \pm \sqrt{\lambda(1 - \sin(\alpha)^2 \lambda)})$, we get $\rho_{W(P_2 P_1)}(\alpha) = \sup_{\lambda \in \sigma(P_2 P_1)} \tilde{v}_1(\lambda, \alpha)$. \square

Remark 3.2. Using a formula due to Lumer [Lum61, Lemma 12], we obtain

$$\begin{aligned} \rho_{W(P_2 P_1)}(\alpha) &= \sup \operatorname{Re}(W(\exp(-i\alpha) P_2 P_1)) \\ &= \lim_{t \rightarrow 0^+} \frac{\|I - t \operatorname{Re}(\exp(-i\alpha) P_2 P_1)\| - 1}{t} \\ &= \lim_{t \rightarrow 0^+} \frac{\|I - t \exp(-i\alpha) P_2 P_1\| - 1}{t}. \end{aligned}$$

In order to make the formula of $\overline{W(P_2 P_1)}$ more explicit, we will describe it as the convex hull of ellipses $\mathcal{E}(\lambda)$. Recall that for $\lambda \in [0, 1]$, $\mathcal{E}(\lambda)$ denote the domain delimited by the ellipse with foci 0 and λ , and minor axis length $\sqrt{\lambda(1 - \lambda)}$. Several of these ellipses are represented in Figure 1.

Remark 3.3. Other descriptions for $\mathcal{E}(\lambda)$ are possible. The Cartesian equation of the boundary of $\mathcal{E}(\lambda)$ is given by:

$$\frac{(x_\lambda - \frac{\lambda}{2})^2}{\frac{\lambda}{4}} + \frac{y_\lambda^2}{\frac{\lambda(1-\lambda)}{4}} = 1,$$

while the parametric equation of the boundary of $\mathcal{E}(\lambda)$ is given by:

$$x_\lambda(t) = \frac{\sqrt{\lambda}}{2} \cos(t) + \frac{\lambda}{2}, \quad y_\lambda(t) = \frac{\sqrt{\lambda(1 - \lambda)}}{2} \sin(t).$$

Lemma 3.4. *Let $\lambda \in [0, 1]$. The support function of the ellipse $\mathcal{E}(\lambda)$ is:*

$$\rho_{\mathcal{E}(\lambda)}(\alpha) = \frac{1}{2}(\cos(\alpha)\lambda + \sqrt{\lambda(1 - \sin(\alpha)^2 \lambda)}).$$

Proof. Let $\lambda \in [0, 1]$. The support function of $\mathcal{E}(\lambda)$ relative to the point 0 is given by $\rho_{\mathcal{E}(\lambda)}(\alpha) = \sup_{t \in \mathbb{R}} x_\lambda(t) \cos(\alpha) + y_\lambda(t) \sin(\alpha)$, where $x_\lambda(t)$ and $y_\lambda(t)$ are the parametrization of the boundary of $\mathcal{E}(\lambda)$. Let $g = g_{\lambda, \alpha}$ be the function defined by the following formula:

$$g_{\lambda, \alpha}(t) = \frac{\lambda}{2} \cos(\alpha) + \frac{\sqrt{\lambda}}{2} \cos(\alpha) \cos(t) + \frac{\sqrt{\lambda(1-\lambda)}}{2} \sin(\alpha) \sin(t).$$

In order to compute $\rho_{\mathcal{E}(\lambda)}(\alpha)$ we only need to study this function for $\alpha \in [0, \pi]$ because $\mathcal{E}(\lambda)$ has $y = 0$ as a symmetry axis.

Suppose that $\cos(\alpha) \neq 0$. We have $g'_{\lambda, \alpha}(t_0) = 0$ if and only if $\tan(t_0) = \sqrt{1-\lambda} \tan(\alpha)$.

So the critical points of $g_{\lambda, \alpha}$ are $t_0 = \arctan(\sqrt{1-\lambda} \tan(\alpha))$ and $t_1 = \arctan(\sqrt{1-\lambda} \tan(\alpha)) + \pi$. We denote $\epsilon_0 = 1, \epsilon_1 = -1$. Using standard trigonometric identities, we get

$$\cos(t_i) = \epsilon_i \frac{1}{\sqrt{1 + (1-\lambda) \tan(\alpha)^2}}, \sin(t_i) = \epsilon_i \frac{\sqrt{1-\lambda} \tan(\alpha)}{\sqrt{1 + (1-\lambda) \tan(\alpha)^2}}.$$

We denote $\epsilon_\alpha = \frac{\cos(\alpha)}{|\cos(\alpha)|}$. Using again some trigonometry formulas, we have:

$$\begin{aligned} 2g_{\lambda, \alpha}(t_i) &= \lambda \cos(\alpha) + \epsilon_i \sqrt{\lambda} \cos(\alpha) \frac{1}{\sqrt{1 + (1-\lambda) \tan(\alpha)^2}} \\ &\quad + \epsilon_i \sqrt{\lambda(1-\lambda)} \sin(\alpha) \frac{\sqrt{1-\lambda} \tan(\alpha)}{\sqrt{1 + (1-\lambda) \tan(\alpha)^2}} \\ &= \lambda \cos(\alpha) + \epsilon_i \epsilon_\alpha \sqrt{\lambda} \sqrt{1-\lambda} \sin(\alpha)^2. \end{aligned}$$

We finally obtain that

$$\rho_{\mathcal{E}(\lambda)}(\alpha) = \frac{1}{2} \left(\lambda \cos(\alpha) + \sqrt{\lambda} \sqrt{1-\lambda} \sin(\alpha)^2 \right).$$

Suppose now that $\cos(\alpha) = 0$. Then $g_{\lambda, \alpha}(t) = \frac{\sqrt{\lambda(1-\lambda)}}{2} \sin(t)$. So we get that in all situations $\rho_{\mathcal{E}(\lambda)}(\alpha) = \frac{\sqrt{\lambda(1-\lambda)}}{2}$. We obtain $\rho_{\mathcal{E}(\lambda)}(\alpha) = \tilde{v}_1(\lambda, \alpha)$ for every α . \square

Now we can easily prove Theorem 1.2 in the "generic position" case.

Theorem 3.5. *If (N_1, N_2) are in generic position, then:*

$$\overline{W(P_2 P_1)} = \overline{\text{conv}_{\lambda \in \sigma(P_2 P_1)} \{\mathcal{E}(\lambda)\}}.$$

Proof. We first notice that:

$$\rho_{W(P_2 P_1)}(\alpha) = \sup_{\lambda \in \sigma(P_2 P_1)} \tilde{v}_1(\lambda, \alpha) = \sup_{\lambda \in \sigma(P_2 P_1)} \rho_{\mathcal{E}(\lambda)}(\alpha).$$

As the support function characterizes the closure of a convex bounded set, we simply use Lemma 2.5 to conclude. \square

The proof of the general case follows now by combining the previous theorem with the decomposition (1).

Proof of Theorem 1.2. Recall that $M_1 \neq H$ or $M_2 \neq H$. We use the notation of the orthogonal decomposition (1) of H . Suppose that $\tilde{H} = \{0\}$. Then $P_{M_2} P_{M_1}$ is the direct sum of 0 and I (or is zero if $M_1 \cap M_2 = \{0\}$). Then it is easy to see that $\mathcal{E}(0) = \{0\}$ and $\mathcal{E}(1) = [0, 1]$. So we have $W(P_{M_2} P_{M_1}) = [0, 1] = \text{conv}\{\mathcal{E}(0), \mathcal{E}(1)\}$. When $M_1 \cap M_2 = \{0\}$, we have $W(P_{M_2} P_{M_1}) = \{0\} = \mathcal{E}(0)$.

Suppose $\tilde{H} \neq \{0\}$, and $M_1^\perp \cap M_2^\perp \neq \{0\}$ (the cases $M_1^\perp \cap M_2 \neq \{0\}$ and $M_1 \cap M_2^\perp \neq \{0\}$ are similar). On the space $M_1^\perp \cap M_2^\perp$, we have $P_{M_2}P_{M_1} = 0$. The numerical range of $P_{M_2}P_{M_1}$ on $(M_1^\perp \cap M_2^\perp) \oplus \tilde{H}$ is $\text{conv}\{\{0\}, \text{conv}_{\lambda \in \sigma(P_2P_1)}\{\mathcal{E}(\lambda)\}\}$. As $\mathcal{E}(0) = \{0\} \subset \mathcal{E}(\lambda)$ for all $\lambda \in [0, 1]$, the numerical range of $P_{M_2}P_{M_1}$ on $(M_1^\perp \cap M_2^\perp) \oplus \tilde{H}$ is given by $\text{conv}_{\lambda \in \sigma(P_2P_1)}\{\mathcal{E}(\lambda)\}$.

Suppose $M_1 \cap M_2 \neq \{0\}$. As $P_{M_2}P_{M_1} = I$ on the intersection $M_1 \cap M_2$, the numerical range of $P_{M_2}P_{M_1}$ on $(M_1 \cap M_2) \oplus \tilde{H}$ is $\text{conv}\{\{1\}, \text{conv}_{\lambda \in \sigma(P_2P_1)}\{\mathcal{E}(\lambda)\}\}$. For every $\lambda \in [0, 1]$ we have $0 \in \mathcal{E}(\lambda)$. As $\tilde{H} \neq \{0\}$, the numerical range of $P_{M_2}P_{M_1}$ on $(M_1 \cap M_2) \oplus \tilde{H}$ is $\text{conv}\{[0, 1], \text{conv}_{\lambda \in \sigma(P_2P_1)}\{\mathcal{E}(\lambda)\}\}$. But $\mathcal{E}(1) = [0, 1]$. So, finally, the numerical range of $P_{M_2}P_{M_1}$ on $(M_1 \cap M_2) \oplus \tilde{H}$ is $\text{conv}_{\lambda \in \sigma(P_{M_2}P_{M_1})}\{\mathcal{E}(\lambda)\}$. This proves the theorem. \square

In the case when $P_{M_1} = I$ and $P_{M_2} = I$, we have of course that $W(P_{M_2}P_{M_1}) = \{1\}$.

Remark 3.6. In [CM11], Corach and Maestripieri proved that the Moore-Penrose pseudoinverse of a product of two orthogonal projections is idempotent (possibly unbounded). Conversely, the Moore-Penrose pseudoinverse of an idempotent is a product of two orthogonal projections. It is well known that the numerical range of a (bounded) idempotent is an ellipse (see [SS10]). By using Halmos' theorem in a similar way as before, it is possible to prove that the closure of numerical range of an idempotent E is the convex hull of the domains delimited by the ellipses $\mathcal{E}^+(\lambda)$ of foci 0, 1 and of minor axis length $\sqrt{\frac{1-\lambda}{\lambda}}$, for λ describing the spectrum $\sigma(E^+)$ of the Moore-Penrose pseudoinverse E^+ of E , i.e.:

$$\overline{W(E)} = \text{conv}_{\lambda \in \sigma(E^+)}\{\mathcal{E}^+(\lambda)\}.$$

As $\mathcal{E}^+(\lambda_1) \subset \mathcal{E}^+(\lambda_2)$, if $\lambda_1 \leq \lambda_2$, the convex hull of all these ellipses will be just the biggest one, and we find another proof that $W(E)$ is an ellipse.

3.2 $W(P_2P_1)$ when $P_1P_2P_1$ is diagonalizable

Let (N_1, N_2) be a pair of closed subspaces of H . Denote $P_i = P_{N_i}$. Suppose that (N_1, N_2) is in generic position. The same proof as before will allow us to deduce the general case from this particular one.

In this section we always assume for simplification that H is separable and make the hypothesis that $P_1P_2P_1$ is diagonalizable, according to the following definition.

Definition 3.7. We say that $P_1P_2P_1$ is *diagonalizable* if there exists an orthonormal basis $(\tilde{h}_n)_{n \in \mathbb{N}}$ of H and a sequence of scalars $(\tilde{\lambda}_n)_{n \in \mathbb{N}}$ such that:

$$P_1P_2P_1x = \sum_{n \in \mathbb{N}} \tilde{\lambda}_n \langle x, \tilde{h}_n \rangle \tilde{h}_n \quad (x \in H).$$

This happens for instance when P_2P_1 is a compact operator. Using our diagonalizability assumption, it will be possible to decompose P_2P_1 as a direct sum of 2×2 matrices. As we know that the numerical range of such a matrix is an ellipse, this will permit to deduce the numerical range of P_2P_1 . We first notice that $0 \leq P_1P_2P_1 \leq I$. Therefore $0 \leq \tilde{\lambda}_n \leq 1$. The next lemma characterizes when $\tilde{h}_n \in N_1$.

Lemma 3.8. *Suppose that (N_1, N_2) is in generic position. We have:*

1. $\tilde{h}_n \in N_1 \Leftrightarrow \tilde{\lambda}_n \neq 0$
2. $\tilde{h}_n \in N_1^\perp \Leftrightarrow \tilde{\lambda}_n = 0$.

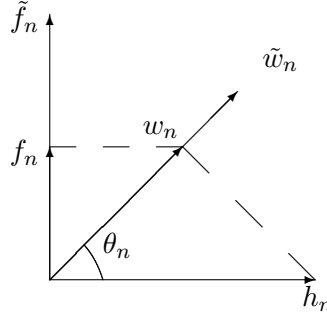


Figure 2:

Proof. We know that $P_1 P_2 P_1 \tilde{h}_n = \tilde{\lambda}_n \tilde{h}_n$. If $\tilde{\lambda}_n \neq 0$, then $\tilde{h}_n = \frac{1}{\tilde{\lambda}_n} P_1 P_2 P_1 \tilde{h}_n \in N_1$. If $\tilde{\lambda}_n = 0$, then $P_1 P_2 P_1 \tilde{h}_n = 0$. So $P_2 P_1 \tilde{h}_n \in N_1^\perp \cap N_2 = \{0\}$, because we are in generic position. So $P_2 P_1 \tilde{h}_n = 0$. We get $P_1 \tilde{h}_n \in N_2^\perp \cap N_1 = \{0\}$, $P_1 \tilde{h}_n = 0$ and thus $\tilde{h}_n \in N_1^\perp$. \square

From now on, we just need those vectors \tilde{h}_n which are in N_1 . For simplification, we denote these vectors as $(h_n)_{n \in \mathbb{N}}$; each one correspond to a nonzero λ_n . This means that $P_1 P_2 P_1 h_n = \lambda_n h_n$. As we have $h_n \in N_1$, we get $P_1 h_n = h_n$. We denote (see Figure 2)

$$w_n = P_2 h_n, \tilde{w}_n = \frac{w_n}{\|w_n\|}, f_n = (I - P_1) P_2 h_n, \tilde{f}_n = \frac{f_n}{\|f_n\|}.$$

Lemma 3.9. *We have $\langle w_n, w_k \rangle = \delta_{n,k} \lambda_n$ and $\langle w_n, h_k \rangle = \delta_{n,k} \lambda_n$, where $\delta_{n,k}$ is the Kronecker symbol, whose value is 1 if $n = k$, and 0 otherwise.*

Proof. For the first equality, we have that $\langle w_n, w_k \rangle = \langle P_2 P_1 h_n, P_2 P_1 h_k \rangle = \langle P_1 P_2 P_1 h_n, h_k \rangle = \lambda_n \langle h_n, h_k \rangle = \delta_{n,k} \lambda_n$. For the other one, we have $\langle w_n, h_k \rangle = \langle P_2 P_1 h_n, P_1 h_k \rangle = \langle P_1 P_2 P_1 h_n, h_k \rangle = \lambda_n \langle h_n, h_k \rangle = \delta_{n,k} \lambda_n$. \square

Corollary 3.10. *Let $\overline{\text{span}}\{h, w\}$ be the closed subspace of H generated by h and w . If $n \neq k$, then $\overline{\text{span}}\{h_n, w_n\}$ is orthogonal to $\overline{\text{span}}\{h_k, w_k\}$.*

Proposition 3.11. *The range of $\overline{\text{span}}\{h_n, w_n\}$ by $P_2 P_1$ verifies*

$$P_2 P_1(\overline{\text{span}}\{h_n, w_n\}) = \overline{\text{span}}\{w_n\} \subset \overline{\text{span}}\{h_n, w_n\}.$$

Proof. We just need to prove that $P_2 P_1(h_n)$ and $P_2 P_1(w_n)$ are collinear with w_n . We have $P_2 P_1(h_n) = w_n$. As h_n is an eigenvector of $P_1 P_2 P_1$, we obtain $P_2 P_1(w_n) = P_2 P_1 P_2 P_1(h_n) = P_2(\lambda_n h_n) = \lambda_n w_n$. \square

Lemma 3.12. *We have $\overline{\text{span}}\{h_n, w_n\} = \overline{\text{span}}\{h_n, f_n\}$.*

Proof. As both of them are subspaces of dimension 2, it will be enough to show that $\overline{\text{span}}\{h_n, w_n\} \subset \overline{\text{span}}\{h_n, f_n\}$. As $h_n \in \overline{\text{span}}\{h_n, f_n\}$, we just need to prove that $w_n \in \overline{\text{span}}\{h_n, f_n\}$. We have $w_n = P_2 P_1 h_n = P_1 P_2 P_1 h_n + (I - P_1) P_2 P_1 h_n = \lambda_n h_n + f_n$. So $w_n \in \overline{\text{span}}\{h_n, f_n\}$. \square

Corollary 3.13. *If $n \neq k$, then $\overline{\text{span}}\{h_n, f_n\}$ is orthogonal to $\overline{\text{span}}\{h_k, f_k\}$. Moreover,*

$$P_2 P_1(\overline{\text{span}}\{h_n, f_n\}) = \overline{\text{span}}\{w_n\} \subset \overline{\text{span}}\{h_n, f_n\}.$$

Proposition 3.14. *We have $\overline{P_2(N_1)} = N_2$.*

Proof. The inclusion $\overline{P_2(N_1)} \subset N_2$ is obvious. In order to prove that $\overline{P_2(N_1)} \supset N_2$, it is enough to show that $P_2(N_1)^\perp \subset N_2^\perp$. Let $y \in P_2(N_1)^\perp$. Then, for every $x \in N_1$, we have $0 = \langle y, P_2(x) \rangle = \langle P_2(y), x \rangle$. So $P_2(y) \in N_1^\perp$. As $P_2(y) \in N_2$ and $N_1^\perp \cap N_2 = \{0\}$, we obtain $P_2(y) = 0$. So $y \in N_2^\perp$. \square

Corollary 3.15. *The vectors $(\tilde{w}_n)_{n \in \mathbb{N}}$ forms an orthonormal basis of N_2 .*

Proof. We know from Lemma (3.9) that $(\tilde{w}_n)_{n \in \mathbb{N}}$ is an orthonormal system in N_2 . It remains to show that it is a generating system. We notice that the inclusion $P_2(N_1) \subset \overline{\text{span}}\{w_n, n \in \mathbb{N}\}$ implies, using $\overline{P_2(N_1)} = N_2$ and $\overline{\text{span}}\{w_n, n \in \mathbb{N}\} = \overline{\text{span}}\{\tilde{w}_n, n \in \mathbb{N}\} \subset N_2$, that

$$N_2 = \overline{P_2(N_1)} \subset \overline{\text{span}}\{\tilde{w}_n, n \in \mathbb{N}\} \subset N_2,$$

and then $N_2 = \overline{\text{span}}\{\tilde{w}_n, n \in \mathbb{N}\}$. Let us show that $P_2(N_1) \subset \overline{\text{span}}\{w_n, n \in \mathbb{N}\}$. For $x \in N_1$, there exists a sequence (ν_n) such that $x = \sum_n \nu_n h_n$. Therefore $P_2(x) = P_2(\sum_n \nu_n h_n) = \sum_n \nu_n P_2(h_n) = \sum_n \nu_n w_n$. Finally $P_2(x) \in \overline{\text{span}}\{w_n, n \in \mathbb{N}\}$. \square

Similarly, we can also show the following proposition.

Proposition 3.16. *We have $\overline{(I - P_1)(N_2)} = N_1^\perp$. Moreover, $(\tilde{f}_n)_{n \in \mathbb{N}}$ is an orthonormal basis of N_1^\perp .*

Corollary 3.17. *The operator $P_2 P_1$ can be written as a direct sum of 2×2 matrices, i.e.:*

$$P_2 P_1 = \bigoplus_{n \in \mathbb{N}} P_2 P_1|_{\overline{\text{span}}\{h_n, \tilde{f}_n\}}.$$

Proof. As $\tilde{f}_n = \frac{f_n}{\|f_n\|}$, we have $\overline{\text{span}}\{h_n, f_n\} = \overline{\text{span}}\{h_n, \tilde{f}_n\}$, and $P_2 P_1(\overline{\text{span}}\{h_n, \tilde{f}_n\}) \subset \overline{\text{span}}\{h_n, \tilde{f}_n\}$. Also, $\overline{\text{span}}\{h_n, \tilde{f}_n\}$ is orthogonal to $\overline{\text{span}}\{h_k, \tilde{f}_k\}$ whenever $n \neq k$. Moreover, $(f_n)_{n \in \mathbb{N}}$ is an orthonormal basis of N_1^\perp . We can write H as $H = N_1 \oplus N_1^\perp = \overline{\text{span}}\{h_n, n \in \mathbb{N}\} \oplus \overline{\text{span}}\{\tilde{f}_n, n \in \mathbb{N}\} = \bigoplus_n \overline{\text{span}}\{h_n, \tilde{f}_n\}$ which proves the result. \square

Lemma 3.18. *With respect to the orthonormal basis (h_n, \tilde{f}_n) , the restriction of $P_2 P_1$ to its invariant subspace $\overline{\text{span}}\{h_n, \tilde{f}_n\}$ is given by:*

$$P_2 P_1|_{\overline{\text{span}}\{h_n, \tilde{f}_n\}} = \begin{pmatrix} \lambda_n & 0 \\ \sqrt{\lambda_n(1-\lambda_n)} & 0 \end{pmatrix}.$$

Proof. As $\tilde{f}_n \in N_1^\perp$, we have $P_1 \tilde{f}_n = 0$, so $P_2 P_1 \tilde{f}_n = 0$. We can represent $P_2 P_1 h_n$ as: $P_2 P_1 h_n = P_1 P_2 P_1 h_n + (I - P_1) P_2 P_1 h_n = \lambda_n h_n + f_n = \lambda_n h_n + \|f_n\| \tilde{f}_n$.

In order to complete the proof, we have to show that $\|f_n\| = \sqrt{\lambda_n(1-\lambda_n)}$. We have $\|f_n\|^2 = \|(I - P_1) P_2 P_1 h_n\|^2 = \|P_2 P_1 h_n\|^2 - \|P_1 P_2 P_1 h_n\|^2 = \langle P_1 P_2 P_1 h_n, h_n \rangle - \|\lambda_n h_n\|^2 = \lambda_n - \lambda_n^2$. \square

Remark 3.19. As $0 \leq P_1 P_2 P_1 \leq I$, we have $0 \leq \lambda_n \leq 1$ for every n . There exists θ_n such that $0 \leq \theta_n \leq \frac{\pi}{2}$ and $\cos(\theta_n)^2 = \lambda_n$. Now we can rewrite $P_2 P_1|_{\overline{\text{span}}\{h_n, \tilde{f}_n\}}$ as:

$$P_2 P_1|_{\overline{\text{span}}\{h_n, \tilde{f}_n\}} = \begin{pmatrix} \cos(\theta_n)^2 & 0 \\ \cos(\theta_n) \sin(\theta_n) & 0 \end{pmatrix}.$$

This corresponds to the matrix of the composition of two orthogonal projections in the plane, projecting onto two lines of angle θ_n .

Corollary 3.20. *The numerical range $W(P_2 P_1|_{\overline{\text{span}}\{h_n, \tilde{f}_n\}})$ is the ellipse $\mathcal{E}(\lambda_n)$.*

Proof. This is consequence of the classical ellipse lemma for the numerical range of a 2×2 matrix (see for instance [GR97]). \square

The following corollary is a "generic position" version of Theorem 1.3.

Corollary 3.21. *Let (N_1, N_2) be two subspaces in generic position such that $P_1 P_2 P_1$ is diagonalizable, then the numerical range $W(P_2 P_1)$ is the convex hull of the ellipses $\mathcal{E}(\lambda)$ for all the λ 's which are non zero eigenvalues of $P_2 P_1$, i.e.:*

$$W(P_2 P_1) = \text{conv}_{\lambda \in \sigma_p(P_2 P_1) \setminus \{0\}} \{\mathcal{E}(\lambda)\}.$$

Proof. Simply combine the fact that $P_2 P_1 = \bigoplus_{n \in \mathbb{N}} P_2 P_1|_{\overline{\text{span}\{h_n, \tilde{f}_n\}}}$, with the fact that $W(T \oplus S) = \text{conv}\{W(T), W(S)\}$ and $W(P_2 P_1|_{\overline{\text{span}\{h_n, \tilde{f}_n\}}}) = \mathcal{E}(\lambda_n)$. \square

Using the same idea as in the proof of Theorem 1.2, we can deduce Theorem 1.3 from Corollary 3.21.

Example 3.22. There are non-trivial examples where $P_1 P_2 P_1$ admits only 0 as eigenvalue (hence $P_1 P_2 P_1$ is not diagonalizable). Let $T \in \mathcal{B}(L_2([0, 1]))$ be defined by $Tf(x) = xf(x)$. One can easily show that T is an injective positive contraction that has no eigenvalues, with $\text{Ker}(I - T) = \{0\}$ and $\sigma(T) = [0, 1]$. If we set $C = T^{1/2}$ and $S = (I - T)^{1/2}$, we easily see that C and S are injective and positive contractions with no eigenvalues such that $C^2 + S^2 = I$. Moreover, C and S commute. We set $H = L_2([0, 1]) \oplus L_2([0, 1])$ and

$$P_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, P_2 = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix}.$$

Then P_1 and P_2 are orthogonal projections onto subspaces sitting in generic position, and

$$P_1 P_2 P_1 = \begin{pmatrix} C^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Suppose there exist $f \oplus g \in H$ and $\lambda \in \sigma(P_2 P_1)$ such that $P_1 P_2 P_1(f \oplus g) = \lambda(f \oplus g)$. Then $xf(x) = \lambda f(x)$ almost everywhere, and $0 = \lambda g(x)$. This implies that $\lambda = 0$ and $f = 0$. So 0 is the only eigenvalue of $P_1 P_2 P_1$. However, we have $\sigma(P_1 P_2 P_1) = \sigma(T) \cup \{0\} = [0, 1]$.

Remark 3.23. At the end of [Nee99], the author asks if $\|P_2 P_1\|^2$ is an accumulation point of eigenvalues, and if the spectrum $P_2 P_1$ without zero consist only of eigenvalues. The previous example answers these two questions negatively.

3.3 Localization of $W(P_2 P_1)$

First we have this simple consequence of Theorem 1.2.

Corollary 3.24. *Let P_1, P_2 be two orthogonal projections. We have:*

$$\overline{W(P_2 P_1)} \subset \overline{\text{conv}_{\lambda \in [0, 1]} \{\mathcal{E}(\lambda)\}}.$$

Proof. If $P_1 = P_2 = I$ this is clear since $W(I) = \{1\}$. Now suppose that $P_1 \neq I$ or $P_2 \neq I$. We use Theorem 1.2 and the fact that $\sigma(P_2 P_1) \subset [0, 1]$, so we have the inclusion $\text{conv}_{\lambda \in \sigma(P_2 P_1)} \{\mathcal{E}(\lambda)\} \subset \text{conv}_{\lambda \in [0, 1]} \{\mathcal{E}(\lambda)\}$. \square

This corollary says that if we can include $\overline{\text{conv}_{\lambda \in [0, 1]} \{\mathcal{E}(\lambda)\}}$ (see Figure 3) in a subset of \mathbb{C} , then for any pair of projection P_1, P_2 we can include $W(P_2 P_1)$ in the same subset. The next lemma is an example of localization of the numerical range using Corollary 3.24.

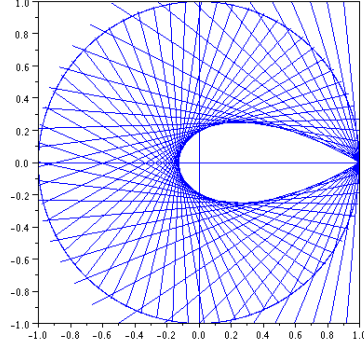


Figure 3: $\text{conv}_{\lambda \in [0,1]} \{\mathcal{E}(\lambda)\}$

Lemma 3.25. *Let P_1 and P_2 be two orthogonal projections. Then $\overline{W(P_2P_1)}$ is a subset of the rectangle whose sides are $x = -\frac{1}{8}$, $x = 1$, $y = \frac{1}{4}$ and $y = -\frac{1}{4}$.*

Proof. Using Corollary 3.24 and the parametric equation of the boundary of $\mathcal{E}(\lambda)$ (see Remark 3.3), we can prove that for all $t \in \mathbb{R}$ and for all $\lambda \in [0, 1]$, we have $-\frac{1}{8} \leq x_\lambda(t) \leq 1$ and $-\frac{1}{4} \leq y_\lambda(t) \leq \frac{1}{4}$. \square

Proof of Proposition 1.5. Suppose that we have found θ_λ such that $\mathcal{E}(\lambda) \subset \{z \in \mathbb{C}, |\arg(1 - z)| \leq \theta_\lambda\}$ for every λ . Taking $\theta = \sup\{\theta_\lambda : \lambda \in \sigma(P_2P_1)\}$, we will have that

$$W(P_2P_1) \subset \text{conv}_{\lambda \in \sigma(P_2P_1)} \{\mathcal{E}(\lambda)\} \subset \{z \in \mathbb{C}, |\arg(1 - z)| \leq \theta\}.$$

First we note that $\mathcal{E}(0) = \{0\}$ and $\mathcal{E}(1) = [0, 1]$. So we have $\theta_0 = \theta_1 = 0$. For $\lambda \in]0, 1[$, we denote $(x_\lambda(t), y_\lambda(t))$ the parametrization of the boundary of $\mathcal{E}(\lambda)$ given in Remark 3.3. We denote $\theta_\lambda(t)$ the angle between the line connecting the points 0 and 1, and the one connecting points 1 and $(x_\lambda(t), y_\lambda(t))$. We have that $\theta_\lambda = \sup_{t \in \mathbb{R}} |\theta_\lambda(t)|$, and

$$\tan(\theta_\lambda(t)) = \frac{y_\lambda(t)}{1 - x_\lambda(t)} = \frac{\sqrt{\lambda(1 - \lambda)} \sin(t)}{2 - \lambda - \sqrt{\lambda} \cos(t)}.$$

By differentiating $\tan(\theta_\lambda(t))$, we can see that t_0 is a critical point if $\cos(t_0) = \frac{\sqrt{\lambda}}{2 - \lambda}$. So we have that

$$\tan(\theta_\lambda) = \frac{\sqrt{\lambda(1 - \lambda)} \sqrt{1 - \frac{\lambda}{(2 - \lambda)^2}}}{2 - \lambda - \sqrt{\lambda} \frac{\sqrt{\lambda}}{(2 - \lambda)}} = \frac{\sqrt{\lambda(1 - \lambda)} \sqrt{(2 - \lambda)^2 - \lambda}}{(2 - \lambda)^2 - \lambda} = \frac{\sqrt{\lambda}}{\sqrt{4 - \lambda}}.$$

As $\theta = \sup_{\lambda \in \sigma(P_2P_1)} \theta_\lambda$, we get that $\tan(\theta) = \sup_{\lambda \in \sigma(P_2P_1) \setminus \{1\}} \frac{\sqrt{\lambda}}{\sqrt{4 - \lambda}}$. Then we conclude using Lemma 2.7. \square

Remark 3.26. We obtain as a consequence the result that the numerical range of a product of two orthogonal projections is included in a sector with vertex 1 and angle $\pi/6$ ([Cro08]). Also, the result of Proposition 1.5 is sharp, in the sense that if $\theta < \arctan(\sqrt{\frac{\cos^2(M_1, M_2)}{4 - \cos^2(M_1, M_2)}})$, then $W(P_2P_1)$ is not included in $\{z \in \mathbb{C}, |\arg(1 - z)| \leq \theta\}$.

3.4 Some examples

Let P_1, P_2 be two orthogonal projections. The spectrum $\sigma(P_2P_1)$ is always a compact subset of $[0, 1]$. In this section, we study the following inverse spectral problem : let K be a compact subset of $[0, 1]$; when two orthogonal projections P_1 and P_2 exist such that $\sigma(P_2P_1) = K$? We will show that the answer is positive if and only if $0 \in K$ or $K = \{1\}$.

We start with the case $K = \{1\}$.

Proposition 3.27. *Let M_1 and M_2 be two subspaces of H . If 0 does not belong to $\sigma(P_{M_2}P_{M_1})$, then we have that $M_1 = M_2 = H$, $P_{M_1} = P_{M_2} = I$ and $\sigma(P_{M_2}P_{M_1}) = \{1\}$.*

Proof. We decompose H as in (1):

$$H = (M_1 \cap M_2) \oplus (M_1 \cap M_2^\perp) \oplus (M_1^\perp \cap M_2) \oplus (M_1^\perp \cap M_2^\perp) \oplus \tilde{H}.$$

Then $P_{M_2}P_{M_1} = I \oplus 0 \oplus 0 \oplus 0 \oplus P_2P_1$. As 0 does not belong to $\sigma(P_2P_1)$, we obtain $M_1 \cap M_2^\perp = M_1^\perp \cap M_2 = M_1^\perp \cap M_2^\perp = \tilde{H} = \{0\}$ (otherwise $P_{M_2}P_{M_1}$ would have a non trivial kernel). So we have $H = M_1 \cap M_2$ and $M_1 = M_2 = H$. Therefore $P_{M_1} = P_{M_2} = I$ and $\sigma(P_{M_2}P_{M_1}) = \sigma(I) = \{1\}$. \square

Now, we suppose that $0 \in K$.

Theorem 3.28. *Let H be a separable Hilbert space. Let K be a compact subset of $[0, 1]$ such that $0 \in K$. Then there exist two orthogonal projections P_1, P_2 on H such that $\sigma(P_2P_1) = K$. Moreover, $P_1P_2P_1$ is diagonalisable.*

Proof. As K is a compact subset of $[0, 1]$, there exists a sequence (λ_n) in K such that $\{\lambda_n, n \in \mathbb{N}\} = K$. For all $n \in \mathbb{N}$, there exists a unique $\theta_n \in [0, \frac{\pi}{2}]$ such that $\lambda_n = \cos(\theta_n)^2$. Let $(e_n)_{n \in \mathbb{N}}$ be an orthonormal basis of H . We denote $h_n = e_{2n}$, $\tilde{f}_n = e_{2n+1}$ and $\tilde{w}_n = \cos(\theta_n)e_{2n} + \sin(\theta_n)e_{2n+1}$. Let $N_1 = \overline{\text{span}}\{h_n, n \in \mathbb{N}\}$ and $N_2 = \overline{\text{span}}\{\tilde{w}_n, n \in \mathbb{N}\}$ (see Figure 2). Then we have that $P_1h_n = h_n$, $P_1\tilde{f}_n = 0$ and $P_2h_n = \cos(\theta_n)^2h_n + \cos(\theta_n)\sin(\theta_n)\tilde{f}_n$, $P_2\tilde{f}_n = \cos(\theta_n)\sin(\theta_n)h_n + \sin(\theta_n)^2\tilde{f}_n$. Hence $P_2P_1h_n = \cos(\theta_n)^2h_n + \cos(\theta_n)\sin(\theta_n)\tilde{f}_n$ and $P_2P_1\tilde{f}_n = 0$. Thus we get

$$\begin{aligned} P_2P_1 &= \bigoplus_{n \in \mathbb{N}} P_2P_1|_{\overline{\text{span}}\{h_n, \tilde{f}_n\}} \\ &= \bigoplus_{n \in \mathbb{N}} \begin{pmatrix} \cos(\theta_n)^2 & 0 \\ \cos(\theta_n)\sin(\theta_n) & 0 \end{pmatrix}. \end{aligned}$$

Also, $\sigma(P_2P_1) = \overline{\{\cos(\theta_n)^2, n \in \mathbb{N}\} \cup \{0\}} = \overline{\{\lambda_n, n \in \mathbb{N}\} \cup \{0\}} = K$. \square

Remark 3.29. We have proved in the previous section that $\overline{W(P_2P_1)} \subset \overline{\text{conv}_{\lambda \in [0, 1]} \{\mathcal{E}(\lambda)\}}$. There are examples where this inclusion is an equality. According to Theorem 1.2, we just need two projections that satisfy $\sigma(P_2P_1) = [0, 1]$. The projections of Example 3.22 satisfy this condition, but $P_1P_2P_1$ is not diagonalisable. With Theorem 3.28, we can also construct an example such that $P_1P_2P_1$ is diagonalisable and $\sigma(P_2P_1) = [0, 1]$.

Remark 3.30. As we now know all the possible shapes of $\sigma(P_2P_1)$, Theorem 1.2 gives all the possible shapes of $\overline{W(P_2P_1)}$.

Remark 3.31. Using the parametrization of the boundary of $\mathcal{E}(\lambda)$ (see Remark 3.3), we can prove that for all $\lambda \in [0, \frac{1}{4}]$, $\mathcal{E}(\lambda) \subset \mathcal{E}(\frac{1}{4})$. Let $K_1 = [0, \frac{1}{4}]$ and $K_2 = \{0, \frac{1}{4}\}$. It

follows from Theorem 3.28 that there exist orthogonal projections P_1, P_2, Q_1, Q_2 such that $\sigma(P_2P_1) = K_1$ and $\sigma(Q_2Q_1) = K_2$. Moreover, we have that:

$$\overline{W(P_2P_1)} = \overline{\text{conv}_{\lambda \in [0, \frac{1}{4}]} \{\mathcal{E}(\lambda)\}} = \mathcal{E}(\frac{1}{4}) = \overline{\text{conv}\{\mathcal{E}(0), \mathcal{E}(\frac{1}{4})\}} = \overline{W(Q_2Q_1)}.$$

This shows that the points of the spectrum of P_2P_1 which are less than $\frac{1}{4}$ are not uniquely determined by the numerical range. We will see in the next section that the situation is different for spectral values greater than $\frac{1}{4}$.

4 The spectrum of P_2P_1 in terms of the numerical range

4.1 The relationship between the spectral and numerical radii

In this section, we will prove proposition 1.4, and compare this result with an inequality from [Kit03].

Proof of Proposition 1.4. If $M_1 = M_2 = H$, this is true. Now we suppose that $M_1 \neq H$ or $M_2 \neq H$. By combining the definition of the numerical radius with the Theorem 1.2, we obtain:

$$\omega(P_2P_1) = \sup_{w \in \overline{W(P_2P_1)}} |w| = \sup_{w \in \mathcal{E}(\lambda), \lambda \in \sigma(P_2P_1)} |w|.$$

First, we compute $\sup_{w \in \mathcal{E}(\lambda)} |w|$ for a fixed λ . We denote by $(x_\lambda(t), y_\lambda(t))$ the parametrization of the boundary of $\mathcal{E}(\lambda)$ given in Remark 3.3. We have $\sup_{w \in \mathcal{E}(\lambda)} |w| = \sup_{t \in \mathbb{R}} \sqrt{x_\lambda(t)^2 + y_\lambda(t)^2}$ and $x_\lambda(t)^2 + y_\lambda(t)^2 = \frac{1}{4}(\lambda^2 \cos(t)^2 + 2\lambda\sqrt{\lambda} \cos(t) + \lambda)$. Therefore

$$\sup_{w \in \mathcal{E}(\lambda)} |w| = \sqrt{\frac{1}{4}(\lambda^2 + 2\lambda\sqrt{\lambda} + \lambda)} = \frac{1}{2}(\lambda + \sqrt{\lambda}).$$

Finally,

$$\omega(P_2P_1) = \sup_{w \in \mathcal{E}(\lambda), \lambda \in \sigma(P_2P_1)} |w| = \sup_{\lambda \in \sigma(P_2P_1)} \frac{1}{2}(\lambda + \sqrt{\lambda}) = \frac{1}{2}(r(P_2P_1) + \sqrt{r(P_2P_1)}).$$

□

Remark 4.1. In [Kit03], Kittaneh proved that for any operator T , we have the following inequality:

$$\omega(T) \leq \frac{1}{2} \left(\|T\| + \|T^2\|^{\frac{1}{2}} \right). \quad (2)$$

Let us compare Proposition 1.4 with Kittaneh's inequality when $T = P_2P_1$. If $M_1 \cap M_2 \neq \{0\}$, then 1 is eigenvalue of P_2P_1 . So $\|P_2P_1\| = \|(P_2P_1)^2\| = 1$, $r(P_2P_1) = 1$ and $\omega(P_2P_1) = 1$. Thus $\omega(P_2P_1) = \frac{1}{2}(\sqrt{r(P_2P_1)} + r(P_2P_1)) = \frac{1}{2}(\|P_2P_1\| + \|(P_2P_1)^2\|^{\frac{1}{2}})$ and in this case, (2) is an equality.

If $M_1 \cap M_2 = \{0\}$, then according to [KW88, Deu01] we have $\|(P_2P_1)^n\| = \cos(M_1, M_2)^{2n-1}$ and $\|P_1P_2P_1\| = \cos(M_1, M_2)^2 = r(P_1P_2P_1) = r(P_2P_1)$. So we have $\omega(P_2P_1) = \frac{1}{2}(\sqrt{r(P_2P_1)} + r(P_2P_1)) = \frac{1}{2}(\cos(M_1, M_2) + \cos(M_1, M_2)^2)$ and also $\frac{1}{2}(\|P_2P_1\| + \|(P_2P_1)^2\|^{\frac{1}{2}}) = \frac{1}{2}(\cos(M_1, M_2) + \cos(M_1, M_2)^{\frac{3}{2}})$. If $\cos(M_1, M_2) < 1$, then $\omega(P_2P_1) < \frac{1}{2}(\|P_2P_1\| + \|(P_2P_1)^2\|^{\frac{1}{2}})$. So in this case, (2) is a strict inequality.

4.2 How to find $\sigma(P_2P_1)$ from $\overline{W(P_2P_1)}$ (and $\overline{W(P_2(I - P_1))}$)

Contrarily to Sections 2.1 and 2.2, where we have described $\overline{W(P_2P_1)}$ in terms of $\sigma(P_2P_1)$, the aim of this section is to obtain information about the spectrum of P_2P_1 from its numerical range. We give an informal idea about how we do this. Denote $g_\alpha(\lambda) = \frac{1}{2}(\cos(\alpha)\lambda + \sqrt{\lambda(1 - \sin(\alpha)^2\lambda)})$; then we have $\rho_{W(P_2P_1)}(\alpha) = \sup_{\lambda \in \sigma(P_2P_1)} g_\alpha(\lambda)$. We will use the support function as a tool to identify if the ellipse $\mathcal{E}(\lambda)$ is in the numerical range. If this is the case, then λ will be in the spectrum. Denote by \mathcal{S} the closure of $\text{conv}_{\lambda \in [0,1]} \{\mathcal{E}(\lambda)\}$. By Corollary 3.24, we have $W(P_2P_1) \subset \mathcal{S}$, so $\sup_{\lambda \in \sigma(P_2P_1)} g_\alpha(\lambda) = \rho_{W(P_2P_1)}(\alpha) \leq \rho_{\mathcal{S}}(\alpha) = \sup_{\lambda \in [0,1]} g_\alpha(\lambda)$. Using the continuity of the function $g_\alpha(\cdot)$ and the compactity of $\sigma(P_2P_1)$, we get the existence of a point $\lambda_0 \in \sigma(P_2P_1)$ such that $\rho_{W(P_2P_1)}(\alpha) = g_\alpha(\lambda_0)$. With this information we are able to find an explicit formula for $\rho_{\mathcal{S}}(\alpha)$. Moreover, we will see that the equality $\rho_{W(P_2P_1)}(\alpha) = \rho_{\mathcal{S}}(\alpha)$ is equivalent to the presence of a unique point λ_0 (depending only on α) in the spectrum of P_2P_1 .

We begin by giving a necessary and sufficient condition such that λ is a critical point of $g_\alpha(\lambda)$.

Lemma 4.2. *Let $\lambda_0 \in]0, 1[$ and $\alpha \in]0, \pi[$. Then λ_0 is a critical point for g_α if and only if we have:*

$$\alpha = 2 \arcsin(\sqrt{1 - \lambda_0 \sin(\alpha)^2}).$$

Proof. We have $g'_\alpha(\lambda) = \frac{1}{2}(\cos(\alpha) + \frac{1-2\lambda \sin(\alpha)^2}{2\sqrt{\lambda(1-\sin(\alpha)^2\lambda)}})$. Thus $g'_\alpha(\lambda) = 0$ if and only if $\sqrt{\lambda} \cos(\alpha) = \frac{1}{2\sqrt{1-\sin(\alpha)^2\lambda}} - \sqrt{1-\lambda \sin(\alpha)^2}$. Denoting $X = \sqrt{1-\sin(\alpha)^2\lambda}$, we have that $g'_\alpha(\lambda) = 0$ if and only if $\sqrt{\frac{1-X^2}{\sin(\alpha)^2}} \cos(\alpha) = \frac{1}{2X} - X$, or, equivalently, if and only if $\cot(\alpha) = \frac{1-2X^2}{2X\sqrt{1-X^2}}$. We denote $X = \sin(\gamma)$ and get that $\cot(\alpha) = \frac{1-2\sin(\gamma)^2}{2\sin(\gamma)\sqrt{1-\sin(\gamma)^2}} = \cot(2\gamma)$. As $\lambda \in [0, 1]$, we have that $X \in [|\cos(\alpha)|, 1]$, and $\gamma \in [\arcsin(|\cos(\alpha)|), \frac{\pi}{2}] \subset [0, \frac{\pi}{2}]$. So $2\gamma \in [0, \pi]$. Therefore $g'_\alpha(\lambda) = 0$ if and only if $\alpha = 2\gamma$, if and only if $\alpha = 2 \arcsin(\sqrt{1 - \lambda_0 \sin(\alpha)^2})$. \square

The next corollary says that the support functions of $W(P_2P_1)$ for $\alpha \in]0, \frac{\pi}{3}]$ do not give us useful information about $\sigma(P_2P_1)$.

Corollary 4.3. *If $\alpha \in [0, \frac{\pi}{3}]$, and $\lambda_0 \in]0, 1[$, then λ_0 is not a critical point of g_α .*

Proof. We just need to check that Lemma 4.2 fails in this case. If $\lambda_0 \in]0, 1[$, then $2 \arcsin(\sqrt{1 - \lambda_0 \sin(\alpha)^2}) \in] \arcsin(|\cos(\alpha)|), \pi[$. If α satisfies the condition of Lemma 4.2, then $\alpha \in] \arcsin(|\cos(\alpha)|), \pi[$. We want to know when we have $\alpha = 2 \arcsin(|\cos(\alpha)|)$. If $\alpha = 2 \arcsin(|\cos(\alpha)|)$, using some trigonometric formulas, we get that $\sin(\alpha) = 2|\cos(\alpha)|\sin(\alpha)$. So $|\cos(\alpha)| = \frac{1}{2}$. If $\alpha = \frac{\pi}{3}$, then $2 \arcsin(|\cos(\alpha)|) = \frac{\pi}{3} = \alpha$. If $\alpha = \frac{2\pi}{3}$, then $2 \arcsin(|\cos(\alpha)|) = \frac{\pi}{3} \neq \alpha$. In other words, $\alpha = 2 \arcsin(|\cos(\alpha)|)$ if and only if $\alpha = \frac{\pi}{3}$. Moreover, if $\alpha \in [0, \frac{\pi}{3}]$, then we have $\alpha < 2 \arcsin(|\cos(\alpha)|)$, so g_α has no critical point on $]0, 1[$. \square

The following proposition says that $\rho_{W(P_2P_1)}(\alpha)$ can give information on $\sigma(P_2P_1)$ if $\alpha \in [\frac{\pi}{3}, \pi]$.

Proposition 4.4. *If $\alpha \in [\frac{\pi}{3}, \pi]$, then the only critical point of g_α is $\lambda_\alpha = \frac{1+\cos(\alpha)}{2\sin(\alpha)^2}$.*

Proof. From Lemma 4.2, we know that λ is a critical point of g_α if and only if $\alpha = 2 \arcsin(\sqrt{1 - \lambda \sin(\alpha)^2})$. Compose by sinus on each side of the equality and use some trigonometric formulas to get that $\sin(\alpha) = 2\sqrt{1 - \lambda \sin(\alpha)^2}\sqrt{\lambda} \sin(\alpha)$. Dividing each

side by $\sin(\alpha)$ and raising to the square, we get that $4\lambda^2 \sin(\alpha)^2 - 4\lambda + 1 = 0$. Therefore if λ is a critical point of g_α , then $\lambda = \frac{1+\cos(\alpha)}{2\sin(\alpha)^2}$ or $\lambda = \frac{1-\cos(\alpha)}{2\sin(\alpha)^2}$. If $\lambda = \frac{1-\cos(\alpha)}{2\sin(\alpha)^2}$, then

$$\begin{aligned} 2 \arcsin(\sqrt{1 - \lambda \sin(\alpha)^2}) &= 2 \arcsin(\sqrt{\frac{1}{2}(1 + \cos(\alpha))}) \\ &= 2 \arcsin(\cos(\frac{\alpha}{2})) \\ &\neq \alpha. \end{aligned}$$

Lemma 4.2 says that λ is not a critical point of g_α . If $\lambda = \frac{1+\cos(\alpha)}{2\sin(\alpha)^2}$, then

$$\begin{aligned} 2 \arcsin(\sqrt{1 - \lambda \sin(\alpha)^2}) &= 2 \arcsin(\sqrt{\frac{1}{2}(1 - \cos(\alpha))}) \\ &= 2 \arcsin(\sin(\frac{\alpha}{2})) \\ &= \alpha. \end{aligned}$$

According to Lemma 4.2, λ is a critical point of g_α . □

Remark 4.5. The condition $\alpha \in [\frac{\pi}{3}, \pi]$ ensures that $\lambda \in [0, 1]$. We remark that

$$\lambda = \frac{1 + \cos(\alpha)}{2\sin(\alpha)^2} = \frac{1}{2(1 - \cos(\alpha))}.$$

If we have $\frac{\pi}{3} \leq \alpha \leq \pi$, then $\frac{1}{4} \leq \frac{1}{2(1-\cos(\alpha))} \leq 1$. So $\lambda \in [\frac{1}{4}, 1]$.

We give now an explicit formula for $\rho_{\mathcal{S}}(\alpha)$.

Corollary 4.6. *The support function of $\mathcal{S} = \overline{\text{conv}_{\lambda \in [0,1]} \{\mathcal{E}(\lambda)\}}$ is given by the following formula:*

$$\rho_{\mathcal{S}}(\alpha) = \begin{cases} \cos(\alpha) & \text{if } \alpha \in [0, \frac{\pi}{3}] \\ \frac{1}{4(1-\cos(\alpha))} & \text{if } \alpha \in [\frac{\pi}{3}, \pi] \end{cases}.$$

Proof. We know that $\rho_{\mathcal{S}}(\alpha) = \max_{\lambda \in [0,1]} g_\alpha(\lambda)$. We proved previously that if $\alpha \in [0, \frac{\pi}{3}]$, then $\rho_{\mathcal{S}}(\alpha) = \max\{g_\alpha(0), g_\alpha(1)\}$ and if $\alpha \in [\frac{\pi}{3}, \pi]$ then $\rho_{\mathcal{S}}(\alpha) = \max\{g_\alpha(0), g_\alpha(\lambda_\alpha), g_\alpha(1)\}$, with $\lambda_\alpha = \frac{1}{2(1-\cos(\alpha))}$. We have that $g_\alpha(0) = 0$ and $g_\alpha(1) = \cos(\alpha)$ and also $g_\alpha(\lambda_\alpha) = \frac{1}{4(1-\cos(\alpha))}$. Now it remains to show that for any $\alpha \in [\frac{\pi}{3}, \pi]$, we have $g_\alpha(\lambda_\alpha) \geq g_\alpha(1)$. As $\frac{1}{4(1-\cos(\alpha))} - \cos(\alpha) = \frac{1-4\cos(\alpha)+4\cos(\alpha)^2}{4(1-\cos(\alpha))} = \frac{(1-2\cos(\alpha))^2}{4(1-\cos(\alpha))}$, and the last term is always positive, we get the announced result. □

Now we have enough material to prove Theorem 1.6.

Proof of Theorem 1.6. Let $\alpha \in [\frac{\pi}{3}, \pi]$. We know that $\rho_{W(P_2 P_1)}(\alpha) = \sup_{\lambda \in \sigma(P_2 P_1)} g_\alpha(\lambda)$. As $\sigma(P_2 P_1)$ is a compact set and g_α is a continuous function, there exists a $\lambda_0 \in \sigma(P_2 P_1)$ such that: $\rho_{W(P_2 P_1)}(\alpha) = \max_{\lambda \in \sigma(P_2 P_1)} g_\alpha(\lambda) = g_\alpha(\lambda_0)$. According to Proposition 4.4, we have $g_\alpha(\lambda_0) = \frac{1}{4(1-\cos(\alpha))}$ if and only if $\lambda_0 = \lambda_\alpha = \frac{1}{2(1-\cos(\alpha))}$.

"1 \Rightarrow 2": If $\rho_{W(P_2 P_1)}(\alpha) = \frac{1}{4(1-\cos(\alpha))} = g_\alpha(\lambda_0)$, then we have $\lambda_0 = \lambda_\alpha = \frac{1}{2(1-\cos(\alpha))}$. As $\lambda_0 \in \sigma(P_2 P_1)$, we get that $\lambda_\alpha \in \sigma(P_2 P_1)$.

"2 \Rightarrow 1": If $\lambda_\alpha \in \sigma(P_2 P_1)$, then we have that:

$$g_\alpha(\lambda_\alpha) \leq \max_{\lambda \in \sigma(P_2 P_1)} g_\alpha(\lambda) \leq \max_{\lambda \in [0,1]} g_\alpha(\lambda) = g_\alpha(\lambda_\alpha).$$

Therefore

$$\rho_{W(P_2P_1)}(\alpha) = \max_{\lambda \in \sigma(P_2P_1)} g_\alpha(\lambda) = g_\alpha(\lambda_\alpha) = \frac{1}{4(1 - \cos(\alpha))}.$$

□

Given α , Theorem 1.6 tells us whether λ_α is in the spectrum or not by looking at the support function of $W(P_2P_1)$ in the direction α . Given λ , the next corollary tell us in which direction α_λ we have to look to know whether λ is in $\sigma(P_2P_1)$ or not.

Corollary 4.7. *Let $\lambda \in [\frac{1}{4}, 1]$. We denote $\alpha_\lambda = \arccos(1 - \frac{1}{2\lambda})$. The following assertions are equivalent:*

1. $\rho_{W(P_2P_1)}(\alpha_\lambda) = \frac{1}{4(1 - \cos(\alpha_\lambda))}$;
2. $\lambda \in \sigma(P_2P_1)$.

Proof. We denote $f : [\frac{\pi}{3}, \pi] \longrightarrow [\frac{1}{4}, 1]$ the function given by $f(\alpha) = \frac{1}{2(1 - \cos(\alpha))}$. The equivalence follows from Theorem 1.6, and the facts that f is bijective with inverse function given by $\lambda \mapsto \arccos(1 - \frac{1}{2\lambda})$. □

The next proposition is a "trick" to deduce most of the spectrum of P_2P_1 from $\sigma(P_2(I - P_1))$. As $P_2(I - P_1)$ is again a product of two orthogonal projections, all the results of this paper apply also to this operator.

Proposition 4.8. *Let $\lambda \neq 0$. If $\lambda \in \sigma(P_2(I - P_1))$, then $1 - \lambda \in \sigma(P_2P_1)$.*

Proof. We decompose H as in (1). Therefore we have

$$H = (M_1 \cap M_2) \oplus (M_1 \cap M_2^\perp) \oplus (M_1^\perp \cap M_2) \oplus (M_1^\perp \cap M_2^\perp) \oplus \tilde{H}$$

and

$$\begin{aligned} P_{M_1} &\sim I \oplus I \oplus 0 \oplus 0 \oplus \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \\ P_{M_2} &\sim I \oplus 0 \oplus I \oplus 0 \oplus \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix} \\ I - P_{M_1} &\sim 0 \oplus 0 \oplus I \oplus I \oplus \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \\ P_{M_2}P_{M_1} &\sim I \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} C^2 & 0 \\ CS & 0 \end{pmatrix} \\ P_{M_2}(I - P_{M_1}) &\sim 0 \oplus 0 \oplus I \oplus 0 \oplus \begin{pmatrix} 0 & CS \\ 0 & S^2 \end{pmatrix} \end{aligned}$$

We remind that $C^2 + S^2 = I$, so we have that $\sigma(S^2) = 1 - \sigma(C^2)$. Moreover $\sigma(P_{M_2}P_{M_1}) = \cup\{\{1\}, \{0\}, \sigma(C^2) \cup \{0\}\}$ and $\sigma(P_{M_2}(I - P_{M_1})) = \cup\{\{0\}, \{1\}, (1 - \sigma(C^2)) \cup \{0\}\}$ depending on whether the corresponding subspaces are not reduced to $\{0\}$.

Let $\lambda \neq 0$ be such that $\lambda \in \sigma(P_{M_2}(I - P_{M_1}))$. Suppose that $\lambda = 1$ and $M_1^\perp \cap M_2 \neq \{0\}$. Then we get that $P_{M_2}P_{M_1} = 0$ on $M_1^\perp \cap M_2$. So $1 - \lambda = 0$ is an eigenvalue of $P_{M_2}P_{M_1}$.

In the other cases, we get that $\lambda \in 1 - \sigma(C^2)$ and $\tilde{H} \neq \{0\}$, hence $1 - \lambda \in \sigma(C^2) \subset \sigma(P_{M_2}P_{M_1})$. □

Example 4.9. There exist orthogonal projections such that $1 - \sigma(P_{M_2}(I - P_{M_1})) \neq \sigma(P_{M_2}P_{M_1})$. We will exhibit an example in $H = \mathbb{C}^3$. Let (e_1, e_2, e_3) be an orthonormal basis of \mathbb{C}^3 . We set $M_1 = \overline{\text{span}}\{e_1\}$ and $M_2 = \overline{\text{span}}\{e_2\}$. Then we get that $M_1 \cap M_2 = \{0\}$, $M_1 \cap M_2^\perp = \overline{\text{span}}\{e_2\}$, $M_1^\perp \cap M_2 = \overline{\text{span}}\{e_1\}$, $M_1^\perp \cap M_2^\perp = \overline{\text{span}}\{e_3\}$ and $\tilde{H} = \{0\}$. So $P_{M_2}P_{M_1} = 0$, $P_{M_2}(I - P_{M_1}) = P_{M_2}$, $\sigma(P_{M_2}P_{M_1}) = \{0\}$ and $\sigma(P_{M_2}(I - P_{M_1})) = \{0, 1\}$.

Remark 4.10. Theorem 1.6 allows us to deduce $\sigma(P_2P_1) \cap [\frac{1}{4}, 1]$ from $\overline{W(P_2P_1)}$. As $I - P_1$ is also an orthogonal projection, we can also deduce $\sigma(P_2(I - P_1)) \cap [\frac{1}{4}, 1]$ from $\overline{W(P_2(I - P_1))}$. Moreover, Proposition 4.8 allows us to deduce $\sigma(P_2P_1) \cap [0, \frac{3}{4}]$ from $\sigma(P_2(I - P_1)) \cap [\frac{1}{4}, 1]$.

In other words, we can deduce $\sigma(P_2P_1)$ from $\overline{W(P_2P_1)}$ and $\overline{W(P_2(I - P_1))}$.

Proposition 4.11. Let P_1, P_2 be two orthogonal projections. If $\alpha \in [0, \frac{\pi}{2}]$, then we have that

$$\rho_{W(P_2P_1)}(\alpha) = r(\text{Re}(\exp(-i\alpha)P_2P_1)) = \|\text{Re}(\exp(-i\alpha)P_2P_1)\| = \omega(\text{Re}(\exp(-i\alpha)P_2P_1)).$$

This proposition is significant because if we know $r(\text{Re}(\exp(-i\alpha)P_2P_1))$ and $r(\text{Re}(\exp(-i\alpha)P_2(I - P_1)))$ for every $\alpha \in [\frac{\pi}{3}, \frac{\pi}{2}]$ then, by using Theorem 1.6 and Proposition 4.8, we can deduce $\sigma(P_2P_1)$.

Proof. Notice that $\text{Re}(\exp(-i\alpha)P_2P_1)$ is an hermitian operator, so $r(\text{Re}(\exp(-i\alpha)P_2P_1)) = \|\text{Re}(\exp(-i\alpha)P_2P_1)\| = \omega(\text{Re}(\exp(-i\alpha)P_2P_1))$ and the highest positive spectral value of $\text{Re}(\exp(-i\alpha)P_2P_1)$ is the highest positive value in the numerical range. In other words, we just need to prove that for all $\alpha \in [0, \frac{\pi}{2}]$, the highest positive spectral value of $\text{Re}(\exp(-i\alpha)P_2P_1)$ is greater than its lowest negative spectral value.

With the notation of Remark 3.1, we have that $\text{Re}(\exp(-i\alpha)P_2P_1) \sim \tilde{v}_1(C^2, \alpha) \oplus \tilde{v}_2(C^2, \alpha)$. We also have that for all $\lambda \in [0, 1]$ and for all $\alpha \in [0, \pi]$, $\tilde{v}_1(\lambda, \alpha) \geq 0$ and $\tilde{v}_2(\lambda, \alpha) \leq 0$. Moreover $\lambda \in \sigma(C^2)$ if and only if $\tilde{v}_1(\lambda, \alpha)$ and $\tilde{v}_2(\lambda, \alpha) \in \sigma(\text{Re}(\exp(-i\alpha)P_2P_1))$. Therefore $|\tilde{v}_1(\lambda, \alpha)| - |\tilde{v}_2(\lambda, \alpha)| = \tilde{v}_1(\lambda, \alpha) + \tilde{v}_2(\lambda, \alpha) = \lambda \cos(\alpha)$. This last term is positive if $\alpha \in [0, \frac{\pi}{2}]$ and negative if $\alpha \in [\frac{\pi}{2}, \pi]$. So $\alpha \in [0, \frac{\pi}{2}]$ implies that $\rho_{W(P_2P_1)}(\alpha) = r(\text{Re}(\exp(-i\alpha)P_2P_1))$. \square

5 Applications to the rate of convergence in the von Neumann-Halperin theorem and to the uncertainty principle

5.1 Applications to the method of alternating projections

Von Neumann proved (cf. [Deu01, Chapter 9]) the following theorem:

Theorem 5.1. Let M_1, M_2 be two closed subspaces of H . Then for every $x \in H$ we have that:

$$\lim_{n \rightarrow \infty} \|(P_{M_2}P_{M_1})^n x - P_{M_1 \cap M_2} x\| = 0.$$

If we set $N_1 = M_1 \cap (M_1 \cap M_2)^\perp$ and $N_2 = M_2 \cap (M_1 \cap M_2)^\perp$, we have that $N_1 \cap N_2 = \{0\}$. In addition, we have

$$(P_{M_2}P_{M_1})^n - P_{M_1 \cap M_2} = (P_{N_2}P_{N_1})^n$$

for every $n \in \mathbb{N}$. Therefore, the study of the convergence of $(P_{M_2}P_{M_1})^n$ to $P_{M_1 \cap M_2}$ reduces to studying the convergence of $(P_{N_2}P_{N_1})^n$ to 0.

If one looks at the speed of convergence of $(P_{N_2}P_{N_1})^n$ to 0, we have the dichotomy that either $(P_{N_2}P_{N_1})^n$ converges linearly to 0, or $(P_{N_2}P_{N_1})^n$ converges *arbitrarily slowly* to 0. We can characterize arbitrarily slow convergence in many ways; see [BDH09, BGM, DH10a, DH10b] and the references therein.

The novelty of the following characterization of arbitrarily slow convergence is in the use of the numerical range of $P_{N_2}P_{N_1}$ in items 6 through 8.

Proposition 5.2. *Let N_1, N_2 be two closed subspaces of H such that $N_1 \cap N_2 = \{0\}$. The following assertions are equivalent:*

1. $(P_{N_2}P_{N_1})^n$ converges arbitrarily slowly to 0
2. $\|P_{N_2}P_{N_1}\| = 1$
3. $N_1^\perp + N_2^\perp$ is not closed
4. $1 \in \sigma(P_{N_2}P_{N_1})$
5. $\cos(N_1, N_2) = 1$
6. $1 \in \overline{W(P_{N_2}P_{N_1})}$
7. there exists a sequence (λ_n) in $[0, 1[$ such that $\lim \lambda_n = 1$ and for every $n \in \mathbb{N}$, $\mathcal{E}(\lambda_n) \subset \overline{W(P_2P_1)}$
8. there exists $\theta < \frac{\pi}{6}$ such that $W(P_{N_2}P_{N_1}) \subset \{z \in \mathbb{C}, |\arg(1 - z)| \leq \theta\}$.

Proof. We refer to [BDH09, BGM] (see also [Deu01, Chapter 9]) for a proof of the equivalences of the first five assertions.

"6 \Rightarrow 2". As $1 \in \overline{W(P_{N_2}P_{N_1})}$, we can find a sequence (x_n) such that $\|x_n\| = 1$ and $\lim_{n \rightarrow \infty} \langle P_{N_2}P_{N_1}x_n, x_n \rangle = 1$. Since we have that

$$\begin{aligned} \langle P_{N_2}P_{N_1}x_n, x_n \rangle &\leq \|P_{N_2}P_{N_1}x_n\| \|x_n\| \\ &\leq \|P_{N_2}P_{N_1}x_n\| \\ &\leq \|P_{N_2}P_{N_1}\| \\ &\leq 1, \end{aligned}$$

we have that $\|P_{N_2}P_{N_1}\| = 1$.

"4 \Rightarrow 6". As $1 \in \sigma(P_{N_2}P_{N_1})$ and $\sigma(P_{N_2}P_{N_1}) \subset \overline{W(P_{N_2}P_{N_1})}$, we have that $1 \in \overline{W(P_{N_2}P_{N_1})}$.

"7 \Rightarrow 6". This is clear as $x_{\lambda_n}(0) = \frac{\sqrt{\lambda_n}}{2} + \frac{\lambda_n}{2} \in \mathcal{E}(\lambda_n) \subset \overline{W(P_2P_1)}$.

"4 \Rightarrow 7". As $N_1 \cap N_2 = \{0\}$, 1 is not an eigenvalue of $P_{N_2}P_{N_1}$. So there exist $\lambda_n \in \sigma(P_{N_2}P_{N_1})$ such that $\lim_n \lambda_n = 1$. The assertion 7 follows from Theorem 1.2.

"5 \Leftrightarrow 8". This is a consequence of Lemma 1.5. \square

Remark 5.3. In the spirit of [BGM], we can extend "1 \Leftrightarrow 6" to a finite number of projection, to obtain the following statement: If P_{N_1}, \dots, P_{N_r} are orthogonal projections such that $\bigcap_{i=1}^r N_i = \{0\}$, then $(P_{N_r} \dots P_{N_1})^n$ converges arbitrarily slowly to 0 if and only if $1 \in \overline{W(P_{N_r} \dots P_{N_1})}$. The proof is similar.

Remark 5.4. The equivalences between items 5 through 8 still hold if we drop the assumption that $N_1 \cap N_2 = \{0\}$.

5.2 Applications to annihilating pairs

In this section we will give new characterizations of annihilating pairs. First we recall the context. We denote by \mathcal{F} the Fourier transform on $L_2(\mathbb{R})$. Let S and Σ be two measurable subsets of \mathbb{R} . We denote by M_g the operator of multiplication by $g \in L_\infty(\mathbb{R})$ (i.e.: $M_g(f) = gf$ for $f \in L_2(\mathbb{R})$). We denote by 1_S the indicator function of the subset S . Set $P_S = M_{1_S}$ and $P_\Sigma = \mathcal{F}^* M_{1_\Sigma} \mathcal{F}$.

Definition 5.5. We say that (S, Σ) is an *annihilating pair* if for every $f \in L_2(\mathbb{R})$ we have:

$$P_S f = P_\Sigma f = f \Rightarrow f = 0.$$

Definition 5.6. We say that (S, Σ) is a *strong annihilating pair* if there exists a constant $c > 0$ depending on S, Σ such that for all $f \in L_2(\mathbb{R})$ we have:

$$\|f\|^2 \leq c \left(\|(I - P_S)f\|^2 + \|(I - P_\Sigma)f\|^2 \right).$$

We want to recall some known facts ([HJ94], and [Len72]) about (strong) annihilating pairs.

Proposition 5.7. *The following assertions are equivalents:*

1. (S, Σ) is an annihilating pair
2. $1 + i \notin W(P_S + iP_\Sigma)$
3. $\text{Ran}(P_S) \cap \text{Ran}(P_\Sigma) = \{0\}$.

Proposition 5.8. *The following assertions are equivalents:*

- a. (S, Σ) is a strong annihilating pair
- b. $1 + i \notin \overline{W(P_S + iP_\Sigma)}$
- c. $\text{Ran}(P_S) \cap \text{Ran}(P_\Sigma) = \{0\}$ and $\cos(P_S, P_\Sigma) < 1$
- d. $\|P_S P_\Sigma\| < 1$
- e. $r(P_S P_\Sigma) < 1$
- f. $1 \notin \sigma(P_S P_\Sigma)$.

The following proposition is a new characterization of annihilating pairs.

Proposition 5.9. *The following assertions are equivalent to the assertions of Proposition 5.7:*

1. (S, Σ) is an annihilating pair
4. $1 \notin W(P_S P_\Sigma)$.

Proof. We have that $1 \in W(P_S P_\Sigma)$ if and only if there exist $h \in H$ such that $\|h\| = 1$ and $\langle P_S P_\Sigma h, h \rangle = 1$. This is equivalent to the existence of some $h \in H$ such that $\|P_S P_\Sigma h\| = \|h\| = 1$. This last assertion is equivalent to (3) in Proposition 5.7. \square

Proposition 5.10. *The following assertions are equivalent to the assertions of Proposition 5.8:*

- a. (S, Σ) is a strong annihilating pair
- g. $1 \notin \overline{W(P_S P_\Sigma)}$
- h. $\omega(P_S P_\Sigma) < 1$
- i. for all $\alpha \in [0, \frac{\pi}{3}]$, $\omega(\text{Re}(\exp(-i\alpha)P_S P_\Sigma)) < \cos(\alpha)$
- j. there exists $\alpha \in [0, \frac{\pi}{3}]$ such that $\omega(\text{Re}(\exp(-i\alpha)P_S P_\Sigma)) < \cos(\alpha)$
- k. there exists $\theta < \frac{\pi}{6}$ such that $W(P_S P_\Sigma) \subset \{z \in \mathbb{C}, |\arg(1 - z)| \leq \theta\} \setminus \{1\}$.

Proof. “ $f \Leftrightarrow g''$ ”. By Theorem 1.2, $1 \in \overline{W(P_S P_\Sigma)}$ if and only if $\mathcal{E}(1) \subset \overline{W(P_S P_\Sigma)}$, if and only if $1 \in \sigma(P_S P_\Sigma)$.

“ $e \Leftrightarrow h''$ ”. This is a direct consequence of Proposition 1.4.

“ $f \Rightarrow i''$ ”. This is a consequence of Corollary 4.3.

“ $i \Rightarrow j''$ ”. This is trivial.

“ $j \Rightarrow f''$ ”. This is a consequence of Corollary 4.3.

“ $c \Leftrightarrow k''$ ”. This consequence of Lemma 1.5, and of the previous Proposition. \square

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References

- [BDH09] Heinz H. Bauschke, Frank Deutsch, and Hein Hundal. Characterizing arbitrarily slow convergence in the method of alternating projections. *Int. Trans. Oper. Res.* 16, no. 4, 413–425., 2009.
- [BGM] Catalin Badea, Sophie Grivaux, and Vladimir Müller. The rate of convergence in the method of alternating projections. *Algebra i Analiz* 23 (2011), no. 3, 1–30; translation in *St. Petersburg Math. J.* 23 (2012), no. 3, 413–434.
- [BGM10] Catalin Badea, Sophie Grivaux, and Vladimir Müller. A generalization of the Friederichs angle and the method of alternating projections. *C. R. Math. Acad. Sci. Paris* 348, no. 1-2, 53–56., 2010.
- [BL10] Catalin Badea and Yuri Lyubich. Geometric, spectral and asymptotic properties of averaged products of projections in Banach spaces. *Studia Math.* 201, no. 1, 21–35., 2010.
- [BS10] A. Bottcher and I.M. Spitkovsky. A gentle guide to the basics of two projections theory. *Linear Algebra Appl.* 432, no. 6, 1412–1459., 2010.
- [CM11] G. Corach and A. Maestripieri. Products of orthogonal projections and polar decomposition. *Linear Algebra Appl.* 434, no. 6, 1594–1609., 2011.
- [Coh07] Guy Cohen. Iterates of a product of conditional expectation operators. *J. Funct. Anal.* 242, no. 2, 658–668., 2007.
- [Cro07] Michel Crouzeix. Numerical Range and functional calculus in Hilbert space. *J. Funct. Anal.* 244, no. 2, 668–690., 2007.
- [Cro08] Michel Crouzeix. A functional calculus based on the numerical range: applications. *Linear Multilinear Algebra* 56, no. 1-2, 81–103., 2008.
- [DD99] Bernard Delyon and François Delyon. Generalization of von Neumann’s spectral sets and integral representation of operators. *Bull. Soc. Math. France* 127, no. 1, 25–41., 1999.
- [Deu01] Frank Deutsch. *Best Approximation in Inner Product Spaces*. Springer-Verlag, New York, 2001.
- [DH10a] Frank Deutsch and Hein Hundal. Slow convergence of sequences of linear operators I, arbitrarily slow convergence. *J. Approx. Theory* 162, no. 9, 1701–1716., 2010.
- [DH10b] Frank Deutsch and Hein Hundal. Slow convergence of sequences of linear operators II, arbitrarily slow convergence. *J. Approx. Theory* 162, no. 9, 1717–1738., 2010.
- [Gal04] A. Galántai. *Projectors and projection methods*. Kluwer Academic Publishers, Boston, MA, 2004.
- [Gal08] A. Galántai. Subspaces, angles and pairs of orthogonal projections. *Linear Multilinear Algebra* 56, no. 3, 227–260., 2008.
- [GR97] Karl E. Gustafson and Duggirala K.M. Rao. *Numerical Range*. Springer, 1997.

- [Hal69] Paul R Halmos. Two Subspaces. *Trans. Amer. Math. Soc.*, 144, 381–389., 1969.
- [HJ94] Victor Havin and Burglind Jorické. *The Uncertainty Principle in Harmonic Analysis*. Springer-Verlag, 1994.
- [Kit03] Fuad Kittaneh. A numerical radius inequality and an estimate for the numerical radius of the Froebenius companion. *Studia Math.* 158, no. 1, 11–17., 2003.
- [KW88] S. Kayalar and H.L. Weinert. Error bounds for the method of alternating projections. *Math. Control Signals Systems* 1, no. 1, 43–59., 1988.
- [Len72] Andrew Lenard. The numerical range of a pair of projection. *J. Functional Analysis* 10 (1972), 410–423., 1972.
- [Lum61] G. Lumer. Semi inner product spaces. *Trans. Amer. Math. Soc.* 100 1961 29–43., 1961.
- [Nee99] Manuela Nees. Products of orthogonal projections as Carleman operators. *Integral Equations Operator Theory* 35, no. 1, 85–92., 1999.
- [NN87] Stuart Nelson and Michael Neumann. Generalisations of the projection method with applications to SOR theory for Hermitian positive semi definite linear system. *Numer. Math.* 51, no. 2, 123–141., 1987.
- [Roc70] R.T. Rockfellar. *Convex Analysis*. Princeton University Press, 1970.
- [SS10] Valeria Simoncini and Daniel B. Szyld. On the field of values of oblique projections. *Linear Algebra Appl.* 433 (2010), no. 4, 810–818., 2010.